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General supersymmetric solutions of five-dimensional supergravity

Jan B. Gutowski

DAMTP, Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WA, U.K.
E-Mail: J.B.Gutowski@damtp.cam.ac.uk

Wafic Sabra

Centre for Advanced Mathematical Sciences and Physics Department
American University of Beirut
Lebanon
E-Mail: ws00@aub.edu.lb

Abstract: The classification of 1/4-supersymmetric solutions of five dimensional gauged supergravity coupled to arbitrary many abelian vector multiplets, which was initiated in [15], is completed. The structure of all solutions for which the Killing vector constructed from the Killing spinor is null is investigated in both the gauged and the ungauged theories and some new solutions are constructed.

Keywords: Supergravity Models.

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1. Introduction

Recently there has been a lot of interest in the study of black holes in gauged supergravity theory. This has been to a large extent motivated by the conjectured equivalence between string theory on anti-de Sitter (AdS) spaces and certain superconformal gauge theories living on the boundary of AdS [1−3]. From the point of view of the dual CFT, supergravity vacua could correspond to an expansion around non-zero vacuum expectation values of certain operators, or describe a holographic renormalization group flow [4]. It is hoped that such equivalence will allow some understanding of the nonperturbative structure of these gauge theories by studying classical supergravity solutions. An example in this direction is the Hawking-Page phase transition [5] which is interpreted as a thermal phase transition from a confining to a deconfining phase in the dual four dimensional $N = 4$ supersymmetric Yang-Mills theory [6].

In this paper we will concentrate on five-dimensional $N = 2$ gauged supergravity coupled to vector multiplets and the classification of their supersymmetric solutions. These are relevant for the holographic descriptions of four dimensional field theories with less than maximal supersymmetry. In the past few years, a lot of effort has been devoted to finding solutions of these theories. For example, magnetically charged string solutions preserving a quarter of supersymmetry were given in [7, 8]. Also, supersymmetric electric solutions preserving half of supersymmetry have been discussed in [9, 10] . However, those electric and rotating solutions have naked singularities or naked closed timelike curves. Some black string solutions and domain wall solutions with non-trivial scalar fields were also presented in [11].

More recently a systematic approach has been used to classify supersymmetric solutions of the minimal gauged five dimensional supergravity [12]. This approach was first used by Tod [13] for the classification of supersymmetric solutions of minimal $\mathcal{N} = 2$, $D = 4$ supergravity. The basic idea in this analysis is to assume the existence of a Killing spinor, (i.e., to assume that the solution preserves at least one supersymmetry) and construct differential forms as bilinears in the Killing spinor. These forms satisfy algebraic and differential conditions, which are sufficient to determine the local form of the metric and the bosonic fields in the theory. In [12], the solutions fall into two classes depending on whether the Killing vector constructed from the Killing spinor is null or time-like. It must be stressed that this general framework can generate many new interesting solutions which are not easily found by employing the usual method of simply guessing an Ansätze. Among other explicit solutions, supersymmetric asymptotically anti-de Sitter black hole solutions were constructed for the minimal supergravity theory in [14] and later generalised to the $U(1)^3$ theory (with three R -charges) in [15]. Further generalizations of these black holes have also recently been found in [16] and [17]. These solutions must have non-vanishing angular momentum and unlike the solutions of [10] do not have naked closed timelike curves. Moreover, in [15], explicit algebraic and differential equations were derived for the case where the Killing vector is time-like and the scalar fields take values in symmetric spaces [18].

In this work, the classification which was initiated in [15] is completed. In particular, we relax the requirement that the scalar manifold should be symmetric. The constraint equations in the case for which the Killing vector is time-like are derived. The structure of all solutions with null Killing vector is also investigated in both gauged and ungauged supergravity theories in five dimensions with some explicit solutions given. We organise our work as follows. Section 2 includes a brief review of the theories of $N = 2$, $D = 5$ gauged supergravity coupled to n abelian vector multiplets, the equations of motion and the general analysis of the algebraic and differential properties of the differential forms constructed from a commuting Killing spinor [15]. In section 3 we will analyse the case where the Killing vector is time-like and where the scalars are not necessarily living in a symmetric space. Section 4 contains the analysis of the solutions with null Killing vector in both gauged and ungauged theories. Solutions are constructed which contain the solutions of [11] as a subclass. We present our conclusions in section 5.

2. Supersymmetric solutions of $N = 2$ supergravity

2.1 $N = 2$ supergravity

The action of $N = 2$, $D = 5$ gauged supergravity coupled to n abelian vector multiplets is [18]

$$S = \frac{1}{16\pi G} \int \left(-{}^5R + 2\chi^2 \mathcal{V} - Q_{IJ} F^I \wedge *F^J + Q_{IJ} dX^I \wedge \star dX^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right), \tag{2.1}$$

where I, J, K take values $1, \ldots, n$ and $F^I = dA^I$. The metric has mostly negative signature. C_{IJK} are constants that are symmetric on IJK; in this paper we shall not assume that the C_{IJK} satisfy the non-linear "adjoint identity" which arises when the scalars lie in a symmetric space; though we will assume that Q_{IJ} is invertible, with inverse Q^{IJ}.

The X^I are scalars which are constrained via

$$\frac{1}{6}C_{IJK}X^I X^J X^K = 1\,. \tag{2.2}$$

We may regard the X^I as being functions of $n-1$ unconstrained scalars ϕ^a. It is convenient to define

$$X_I \equiv \frac{1}{6}C_{IJK}X^J X^K \tag{2.3}$$

so that the condition (2.2) becomes

$$X_I X^I = 1\,. \tag{2.4}$$

In addition, the coupling Q_{IJ} depends on the scalars via

$$Q_{IJ} = \frac{9}{2}X_I X_J - \frac{1}{2}C_{IJK}X^K \tag{2.5}$$

so in particular

$$Q_{IJ}X^J = \frac{3}{2}X_I\,, \qquad Q_{IJ}\partial_a X^J = -\frac{3}{2}\partial_a X_I\,. \tag{2.6}$$

The scalar potential can be written as

$$\mathcal{V} = 9V_I V_J \left(X^I X^J - \frac{1}{2}Q^{IJ}\right), \tag{2.7}$$

where V_I are constants.

For a bosonic background to be supersymmetric there must be a spinor[1] ϵ^a for which the supersymmetry variations of the gravitino and dilatino vanish. For the gravitino this requires

$$\left[\nabla_\mu + \frac{1}{8}X_I(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\gamma^\rho)F^I{}_{\nu\rho}\right]\epsilon^a - \frac{\chi}{2}V_I(X^I\gamma_\mu - 3A^I{}_\mu)\epsilon^{ab}\epsilon^b = 0 \tag{2.8}$$

and for the dilatino it requires

$$\left[\left(\frac{1}{4}Q_{IJ}\gamma^{\mu\nu}F^J{}_{\mu\nu} + \frac{3}{4}\gamma^\mu\nabla_\mu X_I\right)\epsilon^a - \frac{3\chi}{2}V_I\epsilon^{ab}\epsilon^b\right]\frac{\partial X^I}{\partial\phi^a} = 0\,. \tag{2.9}$$

The Einstein equation is

$${}^5R_{\alpha\beta} + Q_{IJ}F^I{}_{\alpha\lambda}F^J{}_\beta{}^\lambda - Q_{IJ}\nabla_\alpha X^I\nabla_\beta X^J - \frac{1}{6}g_{\alpha\beta}\left(4\chi^2\mathcal{V} + Q_{IJ}F^I{}_{\mu\nu}F^{J\mu\nu}\right) = 0\,. \tag{2.10}$$

The Maxwell equations (varying A^I) are

$$d\left(Q_{IJ}\star F^J\right) = -\frac{1}{4}C_{IJK}F^J\wedge F^K\,. \tag{2.11}$$

[1]We use symplectic Majorana spinors. Our conventions are the same as [19].

The scalar equations (varying ϕ^a) are

$$\Big[-d(\star dX_I) + \Big(X_M X^P C_{NPI} - \frac{1}{6}C_{MNI}\Big)(F^M \wedge \star F^N - dX^M \wedge \star dX^N) - \\ - \frac{3}{2}\chi^2 V_M V_N Q^{ML} Q^{NP} C_{LPI}\mathrm{dvol}\Big]\frac{\partial X^I}{\partial \phi^a} = 0\,. \quad (2.12)$$

If a quantity L_I satisfies $L_I \partial_a X^I = 0$ then there must be a function M such that $L_I = M X_I$. This implies that the dilatino equation (2.9) can be simplified to

$$\Big[\Big(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\Big)F^J{}_{\mu\nu}\gamma^{\mu\nu} + \frac{3}{4}\gamma^\mu \nabla_\mu X_I\Big]\epsilon^a + \frac{3\chi}{2}\left(X_I V_J X^J - V_I\right)\epsilon^{ab}\epsilon^b = 0\,, \quad (2.13)$$

and the scalar equation can be written as

$$\begin{aligned} & -d\left(\star dX_I\right) + \Big(\frac{1}{6}C_{MNI} - \frac{1}{2}X_I C_{MNJ}X^J\Big)dX^M \wedge \star dX^N + \\ & + \Big(X_M X^P C_{NPI} - \frac{1}{6}C_{MNI} - 6X_I X_M X_N + \frac{1}{6}X_I C_{MNJ}X^J\Big)F^M \wedge \star F^N + \\ & + 3\chi^2\Big(\frac{1}{2}V_M V_N Q^{ML}Q^{NP}C_{LPI} + X_I Q^{MN}V_M V_N - \\ & \qquad - 2X_I X^M X^N V_M V_N\Big)\mathrm{dvol} = 0\,. \end{aligned} \quad (2.14)$$

2.2 General supersymmetric solutions

Following [19], our strategy for determining the general nature of bosonic supersymmetric solutions is to analyse the differential forms that can be constructed from a (commuting) Killing spinor. We first investigate algebraic properties of these forms, and then their differential properties.

From a single commuting spinor ϵ^a we can construct a scalar f, a 1-form V and three 2-forms $\Phi^{ab} \equiv \Phi^{(ab)}$:

$$f\epsilon^{ab} = \bar{\epsilon}^a \epsilon^b\,, \qquad V_\alpha \epsilon^{ab} = \bar{\epsilon}^a \gamma_\alpha \epsilon^b\,, \qquad \Phi^{ab}_{\alpha\beta} = \bar{\epsilon}^a \gamma_{\alpha\beta}\epsilon^b\,. \quad (2.15)$$

f and V are real, but Φ^{11} and Φ^{22} are complex conjugate and Φ^{12} is imaginary. It is convenient to work with three real two-forms $J^{(i)}$ defined by

$$\Phi^{(11)} = J^{(1)} + iJ^{(2)}\,, \qquad \Phi^{(22)} = J^{(1)} - iJ^{(2)}\,, \qquad \Phi^{(12)} = -iJ^{(3)}\,. \quad (2.16)$$

It will be useful to record some of the algebraic identities that can be obtained from the Fierz identity:

$$\begin{aligned} V_\alpha V^\alpha &= f^2\,, \\ J^{(i)} \wedge J^{(j)} &= -2\delta_{ij} f \star V\,, \\ i_V J^{(i)} &= 0\,, \\ i_V \star J^{(i)} &= -f J^{(i)}\,, \\ J^{(i)}_{\gamma\alpha} J^{(j)\gamma}{}_\beta &= \delta_{ij}\left(f^2 \eta_{\alpha\beta} - V_\alpha V_\beta\right) + \epsilon_{ijk} f J^{(k)}_{\alpha\beta}\,, \end{aligned} \quad (2.17)$$

where $\epsilon_{123} = +1$ and, for a vector Y and p-form A, $(i_Y A)_{\alpha_1,\ldots,\alpha_{p-1}} \equiv Y^\beta A_{\beta\alpha_1,\ldots,\alpha_{p-1}}$. Finally, we have

$$V_\alpha \gamma^\alpha \epsilon^a = f \epsilon^a \tag{2.18}$$

and

$$\Phi^{ab}_{\alpha\beta} \gamma^{\alpha\beta} \epsilon^c = 8 f \epsilon^{c(a} \epsilon^{b)} \,. \tag{2.19}$$

Equation (2.17) implies that V is timelike, null or zero. The final possibility can be eliminated using arguments in [19, 20].

We now turn to the differential conditions that arise because ϵ is a Killing spinor. We differentiate f, V, Φ in turn and use (2.8). Starting with f we find

$$df = -i_V \left(X_I F^I \right) , \tag{2.20}$$

which implies $\mathcal{L}_V f = 0$ where $\mathcal{L}$ denotes the Lie derivative. Next, differentiating V gives

$$D_{(\alpha} V_{\beta)} = 0 \,, \tag{2.21}$$

so V is a Killing vector, and

$$dV = 2 f X_I F^I + X_I \star (F^I \wedge V) + 2\chi V_I X^I J^{(1)} \,. \tag{2.22}$$

Finally, differentiating $J^{(i)}$ gives

$$\begin{aligned} D_\alpha J^{(i)}_{\beta\gamma} = & -\frac{1}{2} X_I \left[2 F^I{}_\alpha{}^\delta \left(\star J^{(i)} \right)_{\delta\beta\gamma} - 2 F^I{}_{[\beta}{}^\delta \left(\star J^{(i)} \right)_{\gamma]\alpha\delta} + \eta_{\alpha[\beta} F^{I\delta\epsilon} \left(\star J^{(i)} \right)_{\gamma]\delta\epsilon} \right] - \\ & - 2\chi V_I X^I \delta^{i1} \eta_{\alpha[\beta} V_{\gamma]} + 3\chi \epsilon^{1ij} V_I \left[A^I{}_\alpha J^{(j)}{}_{\beta\gamma} + \frac{1}{3} X^I (\star J^{(j)})_{\alpha\beta\gamma} \right] , \end{aligned} \tag{2.23}$$

which implies

$$dJ^{(i)} = 3\chi \epsilon^{1ij} V_I \left(A^I \wedge J^{(j)} + X^I \star J^{(j)} \right) \tag{2.24}$$

so $dJ^{(1)} = 0$ but $J^{(2)}$ and $J^{(3)}$ are only closed in the ungauged theory (i.e. when $\chi = 0$). Equation (2.24) implies

$$\mathcal{L}_V J^{(i)} = 3\chi \epsilon^{1ij} \left(i_V (V_I A^I) - V_I X^I f \right) J^{(j)} \,. \tag{2.25}$$

Now consider the effect of a gauge transformation $A^I \to A^I + d\Lambda^I$. The Killing spinor equation is invariant provided the spinor transforms according to

$$\begin{aligned} \epsilon^1 &\to \cos\left(\frac{3\chi V_I \Lambda^I}{2} \right) \epsilon^1 - \sin\left(\frac{3\chi V_I \Lambda^I}{2} \right) \epsilon^2 \\ \epsilon^2 &\to \cos\left(\frac{3\chi V_I \Lambda^I}{2} \right) \epsilon^2 + \sin\left(\frac{3\chi V_I \Lambda^I}{2} \right) \epsilon^1 \,. \end{aligned} \tag{2.26}$$

Under these transformations, $f \to f$, $V \to V$ and $J^{(1)} \to J^{(1)}$, but $J^{(2)} + iJ^{(3)} \to e^{-3i\chi V_I \Lambda^I} (J^{(2)} + iJ^{(3)})$, so $J^{(2,3)}$ are only gauge-invariant in the ungauged theory. We shall choose to work in a gauge in which

$$i_V A^I = f X^I \,. \tag{2.27}$$

In such a gauge we have $\mathcal{L}_V J^{(i)} = 0$.

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To make further progress we will examine the dilatino equation (2.13). Contracting with $\bar{\epsilon}^c$ we obtain

$$\mathcal{L}_V X_I = 0 \tag{2.28}$$

and

$$\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right) F^J{}_{\mu\nu}(J^{(i)})^{\mu\nu} = -\frac{3\chi}{2}\delta^{1i}(X_I V_J X^J - V_I)f\,. \tag{2.29}$$

Next, contracting (2.13) with $\bar{\epsilon}^c\gamma^\sigma$ we find

$$i_V F^J = -d(fX^J)\,, \tag{2.30}$$

which implies that

$$\mathcal{L}_V F^J = 0\,. \tag{2.31}$$

Hence V generates a symmetry of all of the fields. In the gauge (2.27) we also have

$$\mathcal{L}_V A^I = 0\,. \tag{2.32}$$

Contracting (2.13) with $\bar{\epsilon}^c\gamma^\sigma$ we obtain the identity

$$-\left(\frac{1}{4}Q_{IJ}-\frac{3}{8}X_I X_J\right)F^J{}_{\mu\nu}(\star J^{(i)})_\sigma{}^{\mu\nu} = -\frac{3}{4}(J^{(i)})_\sigma{}^\mu\nabla_\mu X_I - \frac{3\chi}{2}\delta^{i1}(X_I V_J X^J - V_I)V_\sigma\,. \tag{2.33}$$

Finally, contracting (2.13) with $\bar{\epsilon}^c\gamma^{\sigma\lambda}$ gives[2]

$$\begin{aligned}\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right)(F^J{}_{\mu\nu}(\star V)^{\sigma\lambda\mu\nu} + 2fF^{J\lambda\sigma}) &= -\frac{3}{4}(\nabla^\lambda X_I V^\sigma - \nabla^\sigma X_I V^\lambda)\\ &\quad + \frac{3\chi}{2}(X_I V_J X^J - V_I)(J^{(1)})^{\sigma\lambda}\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}&\left(\frac{1}{2}Q_{IJ} - \frac{3}{4}X_I X_J\right)\left(F^{J\sigma}{}_\nu(J^{(i)})^{\nu\lambda} - F^{J\lambda}{}_\nu(J^{(i)})^{\nu\sigma}\right) =\\ &\qquad = \frac{3}{4}\nabla_\mu X_I(\star J^{(i)})^{\sigma\lambda\mu} + \frac{3\chi}{2}\epsilon^{1ij}(X_I V_J X^J - V_I)\left(J^{(j)}\right)^{\sigma\lambda}\,.\end{aligned} \tag{2.35}$$

3. The timelike case

As in [19, 12], it is useful to consider two classes of solution, depending on whether the scalar f vanishes everywhere or not. In the null class, the vector V is globally a null Killing vector. In the timelike class, there is some open set $\mathcal{U}$ in which f is non-vanishing and hence V is a timelike Killing vector field, and without loss of generality one can take $f > 0$ in $\mathcal{U}$ [19]. We first analyse the timelike class be examining the constraints imposed by supersymmetry in the region $\mathcal{U}$. This analysis is very similar to that presented in [15] for the case in which the scalars lie in a symmetric space, and the C_{IJK} satisfy an additional non-linear algebraic constraint. We will not assume this here however, unless stated explicitly.

[2]These equations correct some typographical errors found in [15, equations (2.53) and (2.54)], though these equations were not actually used in the analysis of that paper.

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Introduce coordinates (t, x^m) such that $V = \partial/\partial t$. The metric can then be written locally as

$$ds^2 = f^2(dt + \omega)^2 - f^{-1} h_{mn} dx^m dx^n \,. \tag{3.1}$$

The metric h_{mn} can be regarded as the metric on a four dimensional riemannian manifold, which we shall refer to as the "base space" B. ω is a 1-form on B. Since V is Killing, f, ω and h are independent of t. We shall reduce the necessary and sufficient conditions for supersymmetry to a set of equations on B. Let

$$e^0 = f(dt + \omega)\,. \tag{3.2}$$

We choose the orientation of B so that $e^0 \wedge \eta_4$ is positively oriented in five dimensions, where η_4 is the volume form of B. The two form $d\omega$ can be split into self-dual and anti-self-dual parts on B:

$$f d\omega = G^+ + G^- \,, \tag{3.3}$$

where the factor of f is included for convenience.

Equation (2.17) implies that the 2-forms $J^{(i)}$ can be regarded as anti-self-dual 2-forms on the base space;

$$\star_4 J^{(i)} = -J^{(i)} \,, \tag{3.4}$$

where $\star_4$ denotes the Hodge dual on B. Moreover, they also satisfy

$$J^{(i)}{}_m{}^p J^{(j)}{}_p{}^n = -\delta^{ij} \delta_m{}^n + \epsilon_{ijk} J^{(k)}{}_m{}^n \,, \tag{3.5}$$

where indices $m, n, \ldots$ have been raised with h^{mn}, the inverse of h_{mn}. This equation shows that the $J^{(i)}$'s satisfy the algebra of imaginary unit quaternions, i.e., B admits an almost hyper-Kähler structure, just as in [19, 12].

To proceed, we use (2.20) and (2.22) to obtain

$$\begin{aligned} X_I F^I &= de^0 - \frac{2}{3} G^+ - 2\chi f^{-1} V_I X^I J^{(1)} \\ &= -f^{-1} e^0 \wedge df + \frac{1}{3} G^+ + G^- - 2\chi f^{-1} V_I X^I J^{(1)} \,. \end{aligned} \tag{3.6}$$

From (2.23) we find that

$$\begin{aligned} \nabla_m J^{(1)}_{np} &= 0 \\ \nabla_m J^{(2)}_{np} &= P_m J^{(3)}_{np} \\ \nabla_m J^{(3)}_{np} &= -P_m J^{(2)}_{np} \,, \end{aligned} \tag{3.7}$$

where ∇ is the Levi-Civita connection on B and we have defined

$$P_m = 3\chi V_I (A^I{}_m - f X^I \omega_m)\,. \tag{3.8}$$

From (3.5) and (3.7) we conclude that, in the gauged theory, the base space is Kähler, with Kähler form $J^{(1)}$. In the ungauged theory, it is hyper-Kähler with Kähler forms $J^{(i)}$. Again, this is all precisely as in the minimal theories [19, 12].

We are primarily interested in the gauged theory, so we shall assume $\chi \neq 0$. We remark however that the equations constraining the timelike class solutions of the ungauged theory are in fact unchanged from those obtained in [15] (though the constants C_{IJK} were additionally constrained in [15] as the scalars in that paper were assumed to lie on a symmetric space).

Proceeding as in [12], note that we can invert (3.7) to solve for P:

$$P_m = \frac{1}{8}\left(J^{(3)np}\nabla_m J^{(2)}_{np} - J^{(2)np}\nabla_m J^{(3)}_{np}\right), \tag{3.9}$$

from which it follows that

$$dP = \Re, \tag{3.10}$$

where $\Re$ is the Ricci-form of the base space B defined by

$$\Re_{mn} = \frac{1}{2}J^{(1)pq}R_{pqmn} \tag{3.11}$$

and R_{pqmn} denotes the Riemann curvature tensor of B. Hence, once B has been determined, P_m is determined up to a gradient. An argument in [12] shows that the existence of $J^{(2,3)}$ obeying equations (3.5) and (3.7) is a consequence of B being Kähler, and contains no further information.

Next we examine (2.29). It is convenient to write

$$F^I = -f^{-1}e^0 \wedge d(fX^I) + \Psi^I + \Theta^I + X^I G^+, \tag{3.12}$$

where Ψ^I is an anti-self-dual 2-form on B and Θ^I is a self-dual 2-form on B. Equation (3.6) implies

$$X_I\Theta^I = -\frac{2}{3}G^+ \tag{3.13}$$

and

$$X_I\Psi^I = G^- - 2\chi f^{-1}V_I X^I J^{(1)}. \tag{3.14}$$

Now (2.29) determines Ψ^I:

$$\Psi^I = X^I G^- + \frac{3}{2}\chi f^{-1}(Q^{IJ} - 2X^I X^J)V_J J^{(1)} \tag{3.15}$$

hence

$$F^I = d(X^I e^0) + \Theta^I + \frac{3}{2}\chi f^{-1}(Q^{IJ} - 2X^I X^J)V_J J^{(1)}. \tag{3.16}$$

Θ^I is not constrained by the dilatino equation. Finally, from (3.10) together with (3.8) we have the following identity

$$3\chi V_I\Theta^I + \frac{9}{2}\chi^2 f^{-1}(Q^{IJ} - 2X^I X^J)V_I V_J J^{(1)} = \Re. \tag{3.17}$$

Contracting this expression with $J^{(1)}$ we obtain

$$f = \frac{18\chi^2(Q^{IJ} - 2X^I X^J)V_I V_J}{R}, \tag{3.18}$$

where R is the Ricci scalar of B, and hence

$$\Re - \frac{1}{4} R J^{(1)} = 3\chi V_I \Theta^I \,. \tag{3.19}$$

Finally, note that equations (2.18) and (2.19) imply that the spinor obeys the projections

$$\gamma^0 \epsilon^a = \epsilon^a \tag{3.20}$$

and

$$J^{(1)}_{AB} \Gamma^{AB} \epsilon^a = -4 \epsilon^{ab} \epsilon^b \,, \tag{3.21}$$

where indices A, B refer to a vierbein e^A on the base space, and Γ^A are gamma matrices on the base space given by $\Gamma^A = \pm i \gamma^A$. These projections are not independent: (3.21) implies (3.20)

So far we have been discussing constraints on the spacetime geometry and matter fields that are necessary for the existence of a Killing spinor. We shall now argue that these constraints are also sufficient. Assume that we are given a metric of the form (3.1) for which the base space B is Kähler. Let $J^{(1)}$ denote the Kähler form. Assume that f is given in terms of X^I by equation (3.18) and that the field strengths are given by equation (3.16) where Θ^I obeys equations (3.13) and (3.19). Now consider a spinor ϵ^a satisfying the projection (3.21). It is straightforward to show this will automatically satisfy the dilatino equation (2.13). In the basis $(e^0, f^{-1/2} e^A)$, the gravitino equation (2.8) reduces to

$$\partial_t \epsilon^a = 0 \tag{3.22}$$

and

$$\nabla_m \eta^a + \frac{1}{2} P_m \epsilon^{ab} \eta^b = 0 \,, \tag{3.23}$$

where

$$\eta^a = f^{-\frac{1}{2}} \epsilon^a \,. \tag{3.24}$$

The Kähler nature of B guarantees the existence of a solution to equation (3.23) obeying (3.21) without any further algebraic restrictions [21]. Therefore the above conditions on the bosonic fields guarantee the existence of a Killing spinor, i.e., they are both necessary and sufficient for supersymmetry. The only projection required is (3.21), which reduces the number of independent components of the spinor from 8 to 2 so we have at least 1/4 supersymmetry.[3]

We have presented necessary and sufficient conditions for existence of a Killing spinor. However, we are interested in supersymmetric solutions which also satisfy the Bianchi identity $dF^I = 0$ and Maxwell equations (2.11). Substituting the field strengths (3.16) into the Bianchi identities $dF^I = 0$ gives

$$d\Theta^I = -\frac{3}{2} \chi d\big(f^{-1}(Q^{IJ} V_J - 2X^I V_J X^J)\big) \wedge J^{(1)} \,. \tag{3.25}$$

[3] For timelike solutions of the ungauged theory, the only projection that must be imposed on a Killing spinor is (3.20) so the solutions will be 1/2 supersymmetric, as in the minimal theory [19].

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Note that

$$\star F^I = -f^{-2} \star_4 d\left(fX^I\right) + e^0 \wedge \left(\Theta^I + X^I G^+ - \Psi^I\right), \tag{3.26}$$

so the Maxwell equations (2.11) reduce to

$$\begin{aligned} d \star_4 d\left(f^{-1}X_I\right) &= -\frac{1}{6}C_{IJK}\Theta^J \wedge \Theta^K + 2\chi f^{-1}G^- \wedge J^{(1)}V_I + \\ &\quad + \frac{3}{4}\chi^2 f^{-2}\left(C_{IJK}Q^{JM}Q^{KN}V_M V_N + 8V_I V_M X^M\right)\eta_4 , \end{aligned} \tag{3.27}$$

where η_4 denotes the volume form of B.

Finally, the integrability conditions for the existence of a Killing spinor guarantee that the Einstein equation and scalar equations of motion are satisfied as a consequence of the above equations.

In summary, the general timelike supersymmetric solution is determined as follows. First pick a Kähler 4-manifold B. Let $J^{(1)}$ denote the Kähler form and h_{mn} the metric on B. Equation (3.18) determines f in terms of X^I. Next one has to determine ω, X^I and Θ^I by solving equations (3.13), (3.19), (3.25) and (3.27) on B. The metric is then given by (3.1) and the gauge fields by (3.16).

4. The null case

4.1 The general solution

In this section we shall find all solutions of the gauged $N = 2$, $D = 5$ supergravity for which the function f vanishes everywhere.

From (2.22) it can be seen that V satisfies

$$V \wedge dV = 0 \tag{4.1}$$

and is therefore hypersurface-orthogonal. Hence there exist functions u and H such that

$$V = H^{-1}du . \tag{4.2}$$

In addition we find that

$$V \cdot DV = 0 , \tag{4.3}$$

so V is tangent to affinely parameterized geodesics in the surfaces of constant u. One can choose coordinates (u, v, y^m), $m = 1, 2, 3$, such that v is the affine parameter along these geodesics, and hence

$$V = \frac{\partial}{\partial v} . \tag{4.4}$$

The metric must take the form

$$ds^2 = H^{-1}\left(\mathcal{F}du^2 + 2dudv\right) - H^2\gamma_{mn}dy^m dy^n , \tag{4.5}$$

where the quantities H, $\mathcal{F}$, and γ_{mn} depend on u and y^m only (because V is Killing). It is particularly useful to introduce a null basis

$$e^+ = V = H^{-1}du , \qquad e^- = dv + \frac{1}{2}\mathcal{F}du , \qquad e^i = H\hat{e}^i \tag{4.6}$$

satisfying

$$ds^2 = 2e^+e^- - e^ie^i\,, \tag{4.7}$$

where $\hat{e}^i = \hat{e}^i_m dy^m$ is an orthonormal basis for the 3-manifold with u-dependent metric γ_{mn}; $\delta_{ij}\hat{e}^i\hat{e}^j = \gamma_{mn}dy^m dy^n$.

Equation (2.17) implies that $J^{(i)}$ can be written

$$J^{(i)} = e^+ \wedge L^{(i)} \tag{4.8}$$

where $L^{(i)} = L^{(i)}{}_m e^m$ satisfy $L^{(i)}{}_m L^{(j)}{}_n \delta^{mn} = \delta^{ij}$. In fact, by making a change of basis we can set $L^{(i)} = e^i$, so

$$J^{(i)} = e^+ \wedge e^i = du \wedge \hat{e}^i\,. \tag{4.9}$$

For consistency with the computation already done for the minimal theory, we set $\epsilon_{+-123} = -1$. Then (2.24) implies

$$\begin{aligned}
du \wedge d\hat{e}^1 &= 0\\
du \wedge \left[d\hat{e}^2 - 3\chi V_I(A^I \wedge \hat{e}^3 - X^I H\hat{e}^1 \wedge \hat{e}^2)\right] &= 0\\
du \wedge \left[d\hat{e}^3 + 3\chi V_I(A^I \wedge \hat{e}^2 + X^I H\hat{e}^1 \wedge \hat{e}^3)\right] &= 0\,.
\end{aligned} \tag{4.10}$$

Now define $\tilde{d}\hat{e}^i = \frac{1}{2}(\frac{\partial \hat{e}^i_m}{\partial y^n} - \frac{\partial \hat{e}^i_n}{\partial y^m})dy^n \wedge dy^m$. Then (4.10) implies that

$$\begin{aligned}
\tilde{d}\hat{e}^1 &= 0\\
\tilde{d}\hat{e}^2 - 3\chi V_I(A^I \wedge \hat{e}^3 - X^I H\hat{e}^1 \wedge \hat{e}^2) &= 0\\
\tilde{d}\hat{e}^3 + 3\chi V_I(A^I \wedge \hat{e}^2 + X^I H\hat{e}^1 \wedge \hat{e}^3) &= 0\,.
\end{aligned} \tag{4.11}$$

Hence, in particular $(\hat{e}^2 + i\hat{e}^3) \wedge \tilde{d}(\hat{e}^2 + i\hat{e}^3) = 0$ from which it follows that there exists a complex function $S(u,y)$ and real functions $x^2 = x^2(u,y)$, $x^3 = x^3(u,y)$ such that

$$(\hat{e}^2 + i\hat{e}^3)_m = S\frac{\partial}{\partial y^m}(x^2 + ix^3) \tag{4.12}$$

and hence $(\hat{e}^2 + i\hat{e}^3) = Sd(x^2 + ix^3) + \psi du$ for some complex function $\psi(u,y)$. Similarly, there exists a real function $x^1 = x^1(u,y)$ such that $\hat{e}^1 = dx^1 + a_1 du$ for some real function a_1. Hence, from this it is clear that we can change coordinates from u, y^m to u, x^m. Moreover, we can make a gauge transformation of the form $A^I \to A^I + d\Lambda^I$ where $\Lambda^I = \Lambda^I(u,x)$ in order to set $J^{(2)} + iJ^{(3)} \to Sdu \wedge (dx^2 + idx^3)$ where S is now a real function. Note that such a gauge transformation preserves the original gauge restriction (2.27) that $A^I{}_v = 0$.

Hence, the null basis can be simplified to

$$\begin{aligned}
e^+ = V &= H^{-1}du\\
e^- &= dv + \frac{1}{2}\mathcal{F}du\\
e^1 &= H(dx^1 + a_1 du)\\
e^2 &= H(Sdx^2 + S^{-1}a_2 du)\\
e^3 &= H(Sdx^3 + S^{-1}a_3 du)
\end{aligned} \tag{4.13}$$

for real functions $H(u,x^m)$, $S(u,x^m)$, $a_i(u,x^m)$, and $J^{(i)}=e^+\wedge e^i$. We remark that this metric is that of a plane wave, i.e. the supersymmetric Killing vector field V is geodesic and free of expansion, rotation and shear. In order for the geometry to describe a plane-parallel wave we would also require V to be covariantly constant, which corresponds to H being constant.

To proceed, we observe that equation (2.20) implies that $i_V X_I F^I=0$. Moreover, from (2.29) we also find that

$$(\frac{1}{4}Q_{IJ}-\frac{3}{8}X_IX_J)F^J{}_{\mu\nu}(J^i)^{\mu\nu}=0 \tag{4.14}$$

for $i=1,2,3$, which can be rewritten as

$$(\frac{1}{4}Q_{IJ}-\frac{3}{8}X_IX_J)F^J{}_{-i}=0\,. \tag{4.15}$$

Hence, as $X_IF^I{}_{-i}=0$ we find that $F^I{}_{-i}=0$ for $i=1,2,3$. Next, taking (2.35) with $\sigma=-$, $\lambda=i$ we obtain

$$(\frac{1}{4}Q_{IJ}-\frac{3}{8}X_IX_J)F^J{}_{-+}=0 \tag{4.16}$$

and so, recalling that $X_IF^I{}_{-+}=0$, it follows that $F^I{}_{-+}=0$ also. Hence

$$F^I=F^I{}_{+i}e^+\wedge e^i+\frac{1}{2}F^I{}_{ij}e^i\wedge e^j\,. \tag{4.17}$$

Now, from (2.22) we find

$$\begin{aligned}
X_IF^I{}_{12} &= H^{-2}S^{-1}\partial_3H\\
X_IF^I{}_{13} &= -H^{-2}S^{-1}\partial_2H\\
X_IF^I{}_{23} &= H^{-2}\partial_1H-2\chi V_IX^I\,,
\end{aligned} \tag{4.18}$$

where $\partial_i\equiv\partial/\partial x^i$. Next, returning to (2.34) we obtain

$$(\frac{1}{4}Q_{IJ}-\frac{3}{8}X_IX_J)F^J{}_{jk}\epsilon_{ijk}=-\frac{3}{4}\nabla_iX_I+\frac{3\chi}{2}(X_IV_JX^J-V_I)\delta_{i1}\,, \tag{4.19}$$

where $\epsilon_{123}=1$. Hence, it follows from this expression together with (4.18) that

$$\begin{aligned}
F^I{}_{12} &= H^{-2}S^{-1}\partial_3HX^I+H^{-1}S^{-1}\partial_3X^I\\
F^I{}_{13} &= -H^{-2}S^{-1}\partial_2HX^I-H^{-1}S^{-1}\partial_2X^I\\
F^I{}_{23} &= H^{-2}\partial_1HX^I+H^{-1}\partial_1X^I-3\chi Q^{IJ}V_J\,.
\end{aligned} \tag{4.20}$$

On substituting these expressions back into (2.23) we find the following additional constraints

$$\partial_1S=-3\chi V_IX^ISH \tag{4.21}$$

and

$$V_IA^I=V_IA^I{}_udu+\frac{1}{3\chi}S^{-1}(\partial_2Sdx^3-\partial_3Sdx^2) \tag{4.22}$$

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and

$$
\begin{aligned}
X_I F^I = & \left(-2\chi H V_I A^I{}_u + \frac{1}{3} S^{-2} H^{-2} (\partial_2 (H^3 a_3) - \partial_3 (H^3 a_2))\right) du \wedge dx^1 - \\
& - \frac{1}{3} H^{-2} (\partial_1 (H^3 a_3) - \partial_3 (H^3 a_1)) du \wedge dx^2 + \\
& + \frac{1}{3} H^{-2} (\partial_1 (H^3 a_2) - \partial_2 (H^3 a_1)) du \wedge dx^3 + \\
& + (\partial_1 H - 2\chi V_I X^I H^2) S^2 dx^2 \wedge dx^3 - \partial_2 H dx^1 \wedge dx^3 + \partial_3 H dx^1 \wedge dx^2 \,. \quad (4.23)
\end{aligned}
$$

It is particularly convenient to define the "base space" in the null solutions to be the 3-manifold equipped with metric

$$
ds_3^2 = (dx^1)^2 + S^2((dx^2)^2 + (dx^3)^2) \tag{4.24}
$$

with positive orientation fixed by the volume form $dvol_3 = S^2 dx^1 \wedge dx^2 \wedge dx^3$; and we denote the Hodge dual on the base space by $\star_3$. We denote the base metric (4.24) by h_{ij} with inverse h^{ij}, and $\hat{d}$ is the exterior derivative restricted to the base space.

Then F^I can be written as

$$
F^I = du \wedge Y^I + \star_3 \left[\hat{d}(HX^I) - 3\chi H^2 Q^{IJ} V_J dx^1\right] \tag{4.25}
$$

where

$$
Y^I \equiv Y^I{}_1 dx^1 + Y^I{}_2 dx^2 + Y^I{}_3 dx^3 \,. \tag{4.26}
$$

The constraint (4.23) is equivalent to

$$
X_I Y^I = \frac{1}{3} H^{-2} \star_3 \hat{d}(H^3 a) - 2\chi H V_J A^J{}_u dx^1 \,, \tag{4.27}
$$

where $a \equiv a_1 dx^1 + a_2 dx^2 + a_3 dx^3$.

We note that the Bianchi identity implies that

$$
\hat{d} \star_3 \hat{d}(HX^I) = 3\chi S^{-2} \partial_1 (S^2 H^2 Q^{IJ}) V_J \mathrm{dvol}_3 \tag{4.28}
$$

and also

$$
\hat{d} Y^I = \partial_u \big[\star_3 \big(\hat{d}(HX^I) - 3\chi H^2 Q^{IJ} V_J dx^1\big)\big] \,. \tag{4.29}
$$

In addition, the consistency conditions obtained from (4.22) on imposing

$$
d(V_I A^I) = V_I F^I \tag{4.30}
$$

are

$$
V_I Y^I = -\hat{d}(V_I A^I{}_u) + \frac{1}{3\chi} \partial_u \big(\partial_2 \log S dx^3 - \partial_3 \log S dx^2\big) \tag{4.31}
$$

and

$$
\Box \log S = -9\chi^2 H^2 (Q^{IJ} - 2X^I X^J) V_I V_J \,, \tag{4.32}
$$

where $\Box$ denotes the laplacian on the base 3-manifold with metric given by (4.24).

It is also necessary to impose the gauge equation (2.11) from which we find

$$\hat{d}(HQ_{IJ}\star_3 Y^J) = 3\hat{d}(H^3 a)\wedge\left(\frac{1}{2}\hat{d}(H^{-1}X_I)+\chi V_I dx^1\right)+ \\ +\frac{1}{2}C_{IJK}\star_3 Y^J\wedge\left(\hat{d}(HX^K)-3\chi H^2 Q^{KP}V_P dx^1\right). \tag{4.33}$$

Next, we recall that imposing supersymmetry together with the gauge equations implies that all of the components of the Einstein equation are satisfied automatically with the exception of the ++ component, which must be evaluated independently. In particular,

$$R_{++} = \frac{1}{2H}\Box\mathcal{F} + \partial_u\hat{W} - a^i\partial_i\hat{W} - W_{ij}W^{ij} \tag{4.34}$$

where

$$W_{ij} = H\hat{\nabla}_{(i}a_{j)} + (a^k\partial_k H - \partial_u H)h_{ij} - \frac{1}{2}H\partial_u h_{ij} \tag{4.35}$$

and $\hat{W} = W^i{}_i$, indices on W and a are raised with h^{ij}, the inverse of the base metric and $\hat{\nabla}$ denotes the Levi-Civita connection on the base manifold. It is convenient to define 1-forms U^I on the base space by

$$U^I \equiv Y^I + \star_3\left[a\wedge\left(\hat{d}(HX^I) - 3\chi H^2 Q^{IJ}V_J dx^1\right)\right] \tag{4.36}$$

and for any function $g(u,x^i)$ we define

$$\mathcal{D}g \equiv \partial_u g - a^i\partial_i g\,. \tag{4.37}$$

Then $\mathcal{F}$ is constrained via

$$\Box\mathcal{F} = -2H(\mathcal{D}\hat{W} - W_{ij}W^{ij}) + 2HQ_{IJ}\left(U^I_i U^{Ji} + H^2\mathcal{D}X^I\mathcal{D}X^J\right), \tag{4.38}$$

where the indices of U^I have been raised with h^{ij}.

Finally, it remains to substitute the bosonic constraints into the gravitino equation (2.8) and to check that the geometry does indeed admit Killing spinors. If we impose the constraint

$$\gamma^+\epsilon^a = 0 \tag{4.39}$$

on the Killing spinor, then the $\alpha = -$ component of the Killing spinor equation implies that

$$\frac{\partial\epsilon^a}{\partial v} = 0\,, \tag{4.40}$$

so $\epsilon^a = \epsilon^a(u, x^1, x^2, x^3)$. Next we set $\alpha = +$; we find that

$$H\left(\partial_u\epsilon^a - a^i\partial_i\epsilon^a\right) - \frac{\chi}{2}V_I X^I\gamma^-(\gamma^1\epsilon^a + \epsilon^{ab}\epsilon^b) + \frac{3\chi}{2}V_I A^I{}_+(\gamma^1\epsilon^a + \epsilon^{ab}\epsilon^b) = 0\,, \tag{4.41}$$

where we have raised the indices on a_i using the base metric (4.24). Acting on (4.41) with γ^+ we find the algebraic constraint

$$\gamma^1\epsilon^a + \epsilon^{ab}\epsilon^b = 0\,. \tag{4.42}$$

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Next set $\alpha = 1, 2, 3$; it is straightforward to show that these components of the Killing spinor equation imply that

$$\partial_1 \epsilon^a = \partial_2 \epsilon^a = \partial_3 \epsilon^a = 0 \tag{4.43}$$

and substituting this back into (4.41) we also find

$$\partial_u \epsilon^a = 0\,. \tag{4.44}$$

Hence the gravitino equation implies that ϵ^a is constant. Moreover, it is straightforward to check that the dilatino equation (2.9) is satisfied.

It is also useful to examine the effect on the solution of certain co-ordinate transformations. In particular, under the shift $v = v' + g(u, x)$ we note that the form of the solution remains the same, with v replaced by v', and a_i and $\mathcal{F}$ replaced by

$$\begin{aligned} a'_i &= a_i - H^{-3} \nabla_i g \\ \mathcal{F}' &= \mathcal{F} + 2\frac{\partial g}{\partial u} - 2a^i \partial_i g + H^{-3} h^{ij} \partial_i g \partial_j g\,. \end{aligned} \tag{4.45}$$

Hence we see that $H^3 a$ is determined only up to a gradient.

4.2 Solutions with $F^I = 0$

It is particularly instructive to examine the solutions for which $F^I = 0$. To begin, observe that the vanishing of $F^I{}_{12}$ and $F^I{}_{13}$ in (4.20) implies that

$$\partial_2 (H X^I) = \partial_3 (H X^I) = 0 \tag{4.46}$$

and hence

$$\partial_2 H = \partial_3 H = \partial_2 X^I = \partial_3 X^I = 0 \tag{4.47}$$

so H and X^I depend only on x^1 and u. Then, from equation (4.21) it follows that S must be separable as

$$S = S_1(u, x^1) S_2(u, x^2, x^3)\,. \tag{4.48}$$

Note that $V_I Y^I = 0$ implies that $\partial_u \partial_2 \log S_2 = \partial_u \partial_3 \log S_2 = 0$. Hence without loss of generality, we can set $S_2 = S_2(x^2, x^3)$. Moreover, from the vanishing of $V_I F^I$, using (4.22) we find that

$$(\partial_2^2 + \partial_3^2) \log S_2 = 0\,, \tag{4.49}$$

hence by making an appropriate (u and x^1-independent) change of x^2, x^3 co-ordinates together with an u, x^1-independent gauge transformation, we can without loss of generality take $S_2 = 1$ and set $S = S(u, x^1)$. Note that in these new co-ordinates (4.22) also implies that

$$V_I A^I{}_u = P(u) \tag{4.50}$$

for some function $P(u)$.

Next, from the vanishing of $F^I{}_{23}$ in (4.20), we find that

$$\partial_1 (H X^I) = 3\chi Q^{IJ} V_J H^2\,. \tag{4.51}$$

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Contracting this equation with X_I we obtain

$$\partial_1 H = 2\chi X^I V_I H^2 \,. \tag{4.52}$$

Then (4.21) together with (4.52) imply that

$$H = S^{-2/3} Q(u) \tag{4.53}$$

for some function $Q(u)$. However, by making a redefinition of the co-ordinate u we can without loss of generality take $Q = 1$ and so

$$H = S^{-2/3} \,. \tag{4.54}$$

Also, note that (4.51) is equivalent to

$$\partial_1 (H^{-1} X_I) = -2\chi V_I \tag{4.55}$$

which we solve by setting

$$H^{-1} X_I = -2\chi V_I x^1 + \beta_I(u) \tag{4.56}$$

for some functions $\beta_I(u)$. If the scalars lie on a symmetric space, then this equation can be inverted to obtain an explicit solution for H.

Next consider the a_i. By making a co-ordinate transformation of the type given in (4.45) we can set $a_1 = 0$. Then from the constraint (4.27) we find that $H^3 a_2$ and $H^3 a_3$ are independent of x^1. In addition, setting $b_2 = H^3 a_2$, $b_3 = H^3 a_3$ we must satisfy

$$\partial_2 b_3 - \partial_3 b_2 = 6\chi P \,. \tag{4.57}$$

This fixes b_2, b_3 up to an arbitrary gradient of a function of x^2, x^3 and u, this gradient can also be removed by using a co-ordinate transformation of the type given in (4.45). We can therefore set without loss of generality

$$b_2 = -3\chi P x^3, \qquad b_3 = 3\chi P x^2 \,. \tag{4.58}$$

It is convenient to set $x^2 = r\cos\theta$, $x^3 = r\sin\theta$, then the metric can be rewritten as

$$ds^2 = H^{-1}\left[2du\left(dv + \frac{1}{2}\mathcal{F}du\right) - dr^2 - r^2(d\theta + 3\chi P du)^2\right] - H^2 (dx^1)^2 \,. \tag{4.59}$$

So, by making a shift in θ we can without loss of generality set $P = 0$, and the metric can be simplified to

$$ds^2 = H^{-1}\left[2du\left(dv + \frac{1}{2}\mathcal{F}du\right) - (dx^2)^2 - (dx^3)^2\right] - H^2 (dx^1)^2 \,. \tag{4.60}$$

Lastly, it remains to impose the Einstein equations which fix $\mathcal{F}$ via

$$H\partial_1^2 \mathcal{F} + H^4(\partial_2^2 + \partial_3^2)\mathcal{F} - 3\partial_1 H \partial_1 \mathcal{F} = \frac{9}{2} H^6 Q^{IJ} \partial_u \beta_I \partial_u \beta_J \,. \tag{4.61}$$

Hence we see that solutions for which $\mathcal{F} = 0$ must have $\partial_u \beta_I = 0$ and hence H and X^I are also independent of u.

4.3 Magnetic null solutions

In order to find some more general solutions with $F^I \neq 0$ we set

$$F^I = B^I dx^2 \wedge dx^3 \tag{4.62}$$

for some functions B^I, we shall assume that the B^I do not all vanish. From the Bianchi identity we must have $B^I = B^I(x^2, x^3)$. Then it is straightforward to see that for exactly the same reasoning as for the solutions with $F^I = 0$, one must have $H = H(u, x^1)$ and $X^I = X^I(u, x^1)$ with S separable as

$$S = S_1(u, x^1) S_2(x^2, x^3) \tag{4.63}$$

and the vanishing of $V_I Y^I$ implies

$$V_I A^I{}_u = P(u) \tag{4.64}$$

for some function P. The vanishing of Y^I also implies from (4.27) that

$$\frac{1}{3} H^{-2} \star_3 \hat{d}(H^3 a) - 2\chi H P dx^1 = 0\,. \tag{4.65}$$

Next, observe that (4.25) implies that

$$B^I = S^2 \big(\partial_1 (H X^I) - 3\chi H^2 Q^{IJ} V_J\big)\,. \tag{4.66}$$

Then, from the gauge equations (4.33) we find that

$$Q_{IJ} B^J \hat{d}(H^3 a) \wedge dx^1 = 0\,. \tag{4.67}$$

As $Q_{IJ} B^J$ do not all vanish, it follows that

$$\hat{d}(H^3 a) = 0 \tag{4.68}$$

and therefore $P = 0$. By making use of a co-ordinate transformation of the form (4.45) we can also without loss of generality set $a = 0$.

From (4.66) it follows that

$$(S_2)^{-2} B^I = (S_1)^2 (\partial_1 (H X^I) - 3\chi H^2 Q^{IJ} V_J)\,. \tag{4.69}$$

The LHS of this expression is a function of x^2, x^3 alone, whereas the RHS depends on u and x^1 only. Hence we must have

$$B^I = (S_2)^2 q^I \tag{4.70}$$

for some constants q^I. Then the gauge field strengths simplify to

$$F^I = q^I \mathrm{dvol}(M_2) \tag{4.71}$$

where M_2 is the 2-manifold equipped with metric

$$ds^2(M_2) = (S_2)^2 \big((dx^2)^2 + (dx^3)^2\big)\,. \tag{4.72}$$

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Then from (4.22), requiring that $d(V_I A^I) = V_I F^I$ we find that S_2 must satisfy

$$(\partial_2^2 + \partial_3^2) \log S_2 = 3\chi V_I q^I (S_2)^2 \tag{4.73}$$

which is the Liouville equation. Hence, by making a (u, x^1 independent) co-ordinate transformation of x^2, x^3 we can take the metric on M_2 to be that of H^2, $\mathbb{R}^2$ or S^2 according as $\chi V_I q^I > 0$, $\chi V_I q^I = 0$ or $\chi V_I q^I < 0$ respectively.

Also, H, S_1 and X^I are constrained by

$$\partial_1 S_1 = -3\chi V_I X^I S_1 H \tag{4.74}$$

and

$$H^2 (S_1)^2 \partial_1 \big[H^{-1} X_I + 2\chi V_I x^1 \big] = -\frac{2}{3} Q_{IJ} q^J . \tag{4.75}$$

In general, it is not possible to integrate up these equations when $q^I \neq 0$. When the scalar manifold is particularly simple, such as in the "STU" model, solutions (though by no means the most general solution) to these equations have been found [11]. In principle, the equations presented here could be used to find u-dependent generalizations of these solutions; though we shall not pursue this further here.

4.4 Null solutions of the ungauged theory

Having examined the null solutions of the gauged theory, it is straightforward to consider the special case when the gauge parameter vanishes. There is then considerable simplification to many of the equations.

In particular, as the $J^{(i)}$ are all now closed, we can introduce co-ordinates x^i for $i = 1, 2, 3$ such that

$$J^{(i)} = e^+ \wedge e^i = du \wedge dx^i \tag{4.76}$$

and so we can take for a null basis

$$\begin{aligned} e^+ = V &= H^{-1} du \\ e^- &= dv + \frac{1}{2}\mathcal{F} du \\ e^i &= H(dx^i + a_i du) \quad i = 1, 2, 3 . \end{aligned} \tag{4.77}$$

It is then straightforward to show that the differential constraints on the spinor bilinears together with the algebraic constraints on the bilinears obtained from the dilatino equation imply that

$$F^I = du \wedge Y^I + \star_3 \hat{d}(H X^I) , \tag{4.78}$$

where here $\star_3$ denotes the Hodge dual on $\mathbb{R}^3$ equipped with the standard metric

$$ds^2(\mathbb{R}^3) = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{4.79}$$

and positive orientation is defined with respect to the volume form $dx^1 \wedge dx^2 \wedge dx^3$; and $\hat{d}$ is the exterior derivative restricted to $\mathbb{R}^3$. The Y^I are 1-forms on $\mathbb{R}^3$ which must satisfy

$$X_I Y^I = \frac{1}{3} H^{-2} \star_3 \hat{d}(H^3 a) . \tag{4.80}$$

The Bianchi identity implies that

$$\hat{d}Y^I = \star_3 \hat{d}\big(\partial_u(HX^I)\big) \tag{4.81}$$

together with

$$\nabla^2(HX^I) = 0\,, \tag{4.82}$$

where ∇^2 is the laplacian on $\mathbb{R}^3$. Hence we can write

$$X^I = H^{-1}K^I\,, \tag{4.83}$$

where K^I are u-dependent harmonic functions on $\mathbb{R}^3$. H is then fixed in terms of these harmonic functions by

$$H^3 = \frac{1}{6}C_{IMN}K^I K^M K^N\,. \tag{4.84}$$

Next consider the gauge equations, which can be written using the above constraints as

$$C_{IMN}\big(Y^M{}_i\nabla^i K^N + \frac{1}{2}\nabla^i Y^M{}_i K^N\big) = 0\,, \tag{4.85}$$

where here $\nabla_i \equiv \frac{\partial}{\partial x^i}$ and indices are raised with δ^{ij}.

Finally, we note that supersymmetric solutions of the ungauged theory are generically $\frac{1}{2}$-supersymmetric, in contrast to the 1/4 supersymmetric solutions of the gauged theory. It is straightforward to show that on substituting the constraints into the Killing spinor equation we find that

$$\partial_\mu \epsilon^a = 0 \tag{4.86}$$

so that ϵ^a is constant, and is constrained by

$$\gamma^+\epsilon^a = 0\,. \tag{4.87}$$

It is clear that the equations for the ungauged solutions decouple much more straightforwardly than for gauged solutions. In particular, to construct a solution one first chooses u-dependent harmonic functions K^I, and defines H by (4.84) and then X^I are given by (4.83). Now as $\partial_u K^I$ are also harmonic functions on $\mathbb{R}^3$, it follows locally that we can always find a 1-form Y^I on $\mathbb{R}^3$ satisfying (4.81), though this equation only fixes Y^I up to a gradient.

Suppose that $\tilde{Y}^I$ is a particular integral of (4.81); then we can write

$$Y^I = dZ^I + \tilde{Y}^I \tag{4.88}$$

for some functions Z^I. Then (4.85) implies

$$\nabla^2(C_{IMN}Z^M K^N) = -C_{IMN}\big(2\tilde{Y}^M{}_i\nabla^i K^N + \nabla^i\tilde{Y}^M{}_i K^N\big) \tag{4.89}$$

which fixes $C_{IMN}Z^M K^N$ up to some other (u-dependent) harmonic functions on $\mathbb{R}^3$.

Then, given such Y^I, on contracting (4.85) with K^I we obtain the condition

$$\hat{d}\star_3(H^2 X_I Y^I) = 0\,. \tag{4.90}$$

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It follows that there exists $H^3 a$ satisfying (4.80); which is fixed up to an arbitrary gradient. This gradient can be removed by making a shift in the co-ordinate v of the type given in (4.45). Finally, it remains to solve the $++$ component of the Einstein equations which fix $\mathcal{F}$ up to another u-dependent harmonic function on $\mathbb{R}^3$. Hence, it is clear that the whole solution is specified completely in terms of harmonic functions on $\mathbb{R}^3$. Moreover, it is also apparent that there is considerable simplification when the harmonic functions K^I are independent of u; as in this case it follows that H is also independent of u, and one can set $\tilde{Y}^I = 0$ in (4.88) and (4.89). Such solutions were constructed in [22].

5. Conclusions

In this paper we have completed the classification of supersymmetric solutions of $N = 2$, $D = 5$ gauged supergravity which preserve 2 of the 8 supersymmetries. It is known that the only solution which preserves all 8 of the supersymmetries is AdS_5 with vanishing gauge field strengths $F^I = 0$ and constant scalars X^I. It would be interesting to determine the structure of solutions preserving 1/2 or 3/4 of the supersymmetries. One expects the geometries of these solutions to be constrained further by the presence of additional supersymmetry. It has been shown, for a class of solutions in $N = 2$, $D = 4$ supergravity for which the Killing vector obtained from the Killing spinor is null, that there are no 3/4 supersymmetric solutions [23]. The status of 3/4 supersymmetric solutions in the timelike class of the theories examined in [23] has yet to be determined. This situation is in contrast to that encountered in the ungauged theories, where all supersymmetric solutions are either half-supersymmetric or maximally supersymmetric.

There has been significant recent progress in classifying solutions with higher proportions of supersymmetry in various higher-dimensional supergravities. For eleven dimensional supergravity, all solutions preserving $N = 1$ supersymmetry were classified in [24] and [25]; and a large class of solutions preserving $N > 1$ supersymmetry have been classified in [26 − 28]. Also, in [29], solutions preserving various proportions of supersymmetry in a seven-dimensional supergravity were classified. The methods used in these classifications are equivalent in that they all determine types of G-structures associated with certain Killing spinors; this was done for the common sector of type II supergravity in [30]. However, the method used in [26], whereby the spinors are written as differential forms, is particularly interesting, for it seems to allow one to obtain the key geometric constraints in fewer steps than other approaches. This is clearly useful when investigating complicated theories such as type IIB supergravity; and we expect that classifications of solutions of lower-dimensional theories will be significantly simplified by using this method.

The equations which we have obtained in this paper are considerably more complicated than those obtained in the classifications of the minimal gauged and ungauged supergravities. This is not surprising, as solutions of the minimal theory form a very restricted class of solutions when lifted to higher dimensions. Including more multiplets in the lower dimensional theory corresponds to considering more generic solutions in higher dimensions. The constraints obtained from just the Killing spinor equations (not considering the Bianchi or gauge equations) for 1/32 supersymmetric eleven-dimensional solutions in [24, 25] are extremely complicated. Hence we expect solutions of the five-dimensional supergravities with additional vector multiplets included to reflect some of this increased complexity. However, as many eleven-dimensional solutions which are of physical interest have more than 1/32 supersymmetry, it is to be hoped that classifications of these solutions can be obtained which have meaningful geometric properties.

A final outstanding question is whether there are any regular asymptotically AdS black ring solutions. Supersymmetric black rings are known to exist in the ungauged theory [31–35]. As an asymptotically flat black hole can be obtained from these ring solutions by tuning a particular parameter to vanish, it is natural to enquire whether the black hole solutions found in [14] can be regarded as a special case of a family of AdS black rings. It is clear that a better understanding of the near-horizon geometries of solutions with regular horizons could be used to exclude this possibility. Moreover, as there is supersymmetry enhancement near the horizons of ungauged black holes and rings, an examination of 1/2-supersymmetric gauged solutions could be valuable.

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